MINKOWSKI'S SUCCESSIVE MINIMA

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Dedicated to R.P. Bambah on the occasion of his 80th birthday

Abstract. In this note we give a brief survey on some classical and old problems involving Minkowski's successive minima of a 0-symmetric convex body as well as on some recent developments concerning these functionals. Among others we show that the successive minima are closely related to intrinsic volumes, the lattice point enumerator, the zeros of the Ehrhart polynomial and to simultaneous Diophantine approximation problems.

1. Introduction

One of the basic questions in geometry of numbers is to decide whether a given set in the n-dimensional Euclidean space $\mathbb{R}^n$ contains a lattice point of the intgeral lattice $\mathbb{Z}^n$. With respect to the class $\mathcal{K}_0^n$ of all 0-symmetric convex bodies, i.e., convex compact subsets $K \subset \mathbb{R}^n$ with $K = -K$ and $\dim(K) = n$, and the volume functional $\mathrm{vol}(\cdot)$, Minkowski settled this problem [40, pp.75]:

(1.1) *If* $\mathrm{vol}(K) \geq 2^n$ *then* K *contains a non-zero lattice point.*

The n-cube $C_n = \{x \in \mathbb{R}^n : |x_i| \leq 1, 1 \leq i \leq n\}$ of volume 2^n shows that, in general, the constant 2^n can not be replaced by a smaller one. Although the proof of (1.1) is rather simple it has a lot of applications in different branches of mathematics (cf., e.g., [27, pp. 40], [40, pp. 102]). Minkowski also proved an important generalisation of (1.1) for which we have to introduce his successive minima. For $K \in \mathcal{K}_0^n$ and $1 \leq i \leq n$ the i-th successive minimum $\lambda_i(K)$ is defined by

$$\lambda_i(K) = \min\{\lambda > 0 : \dim(\lambda\, K \cap \mathbb{Z}^n) \geq i\},$$

i.e., $\lambda_i(K)\, K$ is the smallest dilate of K containing i linearly independent lattice points. Obviously, $\lambda_i(K) \leq \lambda_{i+1}(K)$ and $\mathrm{int}(\lambda_1(K)\, K) \cap \mathbb{Z}^n = \{0\}$, where int() denotes the interior. Moreover, $\lambda_i(K)$ is homogeneous of degree -1, which means that $\lambda_i(\mu\, K) = (1/\mu)\, \lambda_i(K)$ for any positive number μ. With this notation, (1.1) can be equivalently reformulated as

Theorem 1.1 (Minkowski's 1st theorem on successive minima)**.** *Let* $K \in \mathcal{K}_0^n$*. Then*

$$\mathrm{vol}(K) \leq \left(\frac{2}{\lambda_1(K)}\right)^n. \tag{1.2}$$

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To see that (1.1) and (1.2) are equivalent, we firstly note that for $K \in \mathcal{K}_0^n$ with $\mathrm{vol}(K) \geq 2^n$, (1.2) implies $\lambda_1(K) \leq 1$ and thus K must contain a non-trivial lattice point. On the other hand, since $\mathrm{int}(\lambda_1(K)\, K) \cap \mathbb{Z}^n = \{0\}$, (1.1) shows that $\mathrm{vol}(\lambda_1(K)\, K) \leq 2^n$ and thus we get (1.2). The problem to classify those bodies satisfying (1.2) with equality leads to the theory of extremal bodies (see [27, pp. 82]).

As mentioned above, Minkowski proved a generalisation of (1.2) which is given by the upper bound in the next theorem (cf. [40, pp. 187], [27, pp. 59]).

Theorem 1.2 (Minkowski's 2nd theorem on successive minima)**.** *Let $K \in \mathcal{K}_0^n$. Then*

$$\frac{1}{n!}\prod_{i=1}^{n}\frac{2}{\lambda_i(K)} \leq \mathrm{vol}(K) \leq \prod_{i=1}^{n}\frac{2}{\lambda_i(K)}. \tag{1.3}$$

Both bounds are best possible. For instance, the lower bound is attained by the regular n-cross polytope $C_n^* = \{x \in \mathbb{R}^n : \sum_{i=1}^n |x_i| \leq 1\}$. Here we have $\mathrm{vol}(C_n^*) = 2^n/n!$ and $\lambda_i(C_n^*) = 1$, $1 \leq i \leq n$. Of course, the upper bound is still tight for the cube C_n, but also for arbitrary boxes Q with facets parallel to the coordinate hyperplanes. For instance, let

$$Q = \{x \in \mathbb{R}^n : |x_i| \leq \alpha_i,\ 1 \leq i \leq n\} \tag{1.4}$$

where we may assume $\alpha_i \leq \alpha_{i-1}$. Then $\lambda_i(Q) = 1/\alpha_i$ and thus we have equality in the upper bound. The lower bound is easy to prove. For if, choose n-linearly independent lattice points z_i such that $z_i \in \lambda_i(K)\, K$, $1 \leq i \leq n$. Then K contains the cross polytope $\mathrm{conv}\{\pm(1/\lambda_i(K))\, z_i : 1 \leq i \leq n\}$ and the volume of that cross polytope is at least as large as the left hand side in (1.3). In contrast to the lower bound the upper bound is considered as a rather deep result in geometry of numbers. This is also reflected by the fact that many eminent mathematicians, e.g., Bambah, Woods, Zassenhaus, Davenport, Siegel, Weyl, etc., tried to improve on Minkowski's second theorem and gave themselves interesting proofs (cf., e.g., [3, 15, 50, 55]). For a modern version of Minkowski's original proof we refer to [29].

The main purpose of this note is to give a brief survey on classical problems concerning the successive minima of a 0-symmetric convex body and also to show new developments and further relations of the successive minima to other geometric functionals. We will state the results mainly with respect to the standard lattice $\mathbb{Z}^n$. Most of the results, however, can easily be generalised to arbitrary lattices by a linear transformation.

At this point we also want to remark that Minkowski's theory of successive minima has been applied successfully in a much broader context than presented here; we just mention the fields i) adelic geometry of numbers (cf. [54]), ii) projective toric varieties (cf. [51]) and iii) successive minima of indefinite quadratic forms (cf. [4, 13, 21]).

The paper is organised as follows. In Section 2 we will study some classical generalisation of Minkowski's theorems. Then we want to point out relations between the so called intrinsic volumes and the successive minima. In Section 4 we show that there are also bounds on the number of lattice points contained in a 0-symmetric convex body in terms of the successive minima. For the class of 0-symmetric lattice polytopes there seems to be even some interesting relations between the coefficients of the so called Ehrhart polynomial and the successive minima. This topic is the content of Section 5. Finally, in the last section we show generalisation of Minkowski's fundamental theorems in the context of simultaneous Diophantine approximation problems.

2. Generalisations of Minkowski's theorems

The most natural way to extend (1.1) is to ask for a general lower bound on the the number of lattice points contained in $K \in \mathcal{K}_0^n$ with respect to the volume. Here we have the following result due to van der Corput [14], [27, pp. 51]

Theorem 2.1 (van der Corput)**.** *Let $K \in \mathcal{K}_0^n$. Then*

$$\#(K \cap \mathbb{Z}^n) \geq 2 \left\lfloor \frac{\mathrm{vol}(K)}{2^n} \right\rfloor + 1,$$

where $\lfloor x \rfloor$ denotes the integral part of $x \in \mathbb{R}$. In particular, if $\mathrm{vol}(K) \geq 2^n$ we get (1.1). A generalisation of a different type is due to Siegel [49], [27, pp. 50] which is based on Parseval's identity for multiple Fourier series.

Theorem 2.2 (Siegel)**.** *Let $K \in \mathcal{K}_0^n$ with $\mathrm{int} K \cap \mathbb{Z}^n = \{0\}$. Then*

$$\mathrm{vol}(K) + \frac{\mathrm{vol}(K)}{4^n} \sum_{z \in \mathbb{Z}^n \setminus \{0\}} \left| \int_{(1/2)K} \mathrm{e}^{-2\,\pi \mathrm{i}\, z \cdot x}\, \mathrm{d}x \right|^2 = 2^n,$$

where $x \cdot y$ denotes the usual inner product on $\mathbb{R}^n$. Since the second term on the left hand side is non-negative Theorem 2.2 implies (1.1). In general, however, it is difficult to evaluate the integral above and thus it is hard to take advantage of this beautiful identity of Siegel.

In order to present a direct sharpening of (1.2) we need the notation of the density of a densest lattice packing. For details we refer to [27, pp. 218]. A lattice $\Lambda \subset \mathbb{R}^n$ is called a packing lattice of a set $K \in \mathcal{K}_0^n$ if $\mathrm{int}(2\,K) \cap \Lambda = \{0\}$, i.e., two different lattice translates $a_1 + K$, $a_2 + K$, $a_1 \neq a_2 \in \Lambda$, have no interior points in common. The density $\delta(K, \Lambda)$ of such a non-overlapping lattice arrangement $\Lambda + K$ is the proportion of space which is occupied by the translates $\Lambda + K$, and it is given by

$$\delta(K, \Lambda) = \frac{\mathrm{vol}(K)}{\det \Lambda},$$

where $\det \Lambda$ denotes the determinant of the lattice. The maximum density $\delta(K, \Lambda)$ with respect to all packing lattices Λ of K is called the density of a densest lattice packing of K and it will be denoted by $\delta(K)$.

By the definition of the first successive minimum we know that $\mathbb{Z}^n$ is a packing lattice of the body $(\lambda_1(K)/2)\,K$. Hence we have $\delta(K) \geq \mathrm{vol}((\lambda_1(K)/2)\,K)$ or

$$\mathrm{vol}(K) \leq \delta(K) \left(\frac{2}{\lambda_1(K)} \right)^n. \tag{2.1}$$

Since $\delta(K) \leq 1$ the inequality above may be considered as an improvement of (1.2), which, in particular, takes more into account the shape of the body K. In view of the upper bound in (1.3) it is tempting to improve also (2.1) by replacing $(2/\lambda_1(K))^n$ with the product of the successive minima. This is the content of a famous problem posed by Davenport [16].

Problem 2.3 (Davenport)**.** *Let $K \in \mathcal{K}_0^n$. Is it true that*

$$\operatorname{vol}(K) \leq \delta(K) \prod_{i=1}^{n} \frac{2}{\lambda_i(K)}\,? \tag{2.2}$$

So far it has only been verified for $n = 2$ and for ellipsoids by Minkowski [40, pp. 196], [27, pp. 195], the case $n = 3$ was settled by Woods [58]. His proof is based on Minkowski's classification of of so called critical lattices in dimension 3 which does not seem to be extendable to higher dimensions (cf. [41], [27, pp. 340]).

For the more general class of ray sets of finite type Rogers and independently Chabauty proved an inequality of type (2.2), but with an additional factor of $2^{(1/2)(n-1)}$ on the right hand side. It was shown, however, by Mahler and Chabauty that in this more general setting the additional factor is necessary (cf. [27, pp. 188]).

There are various attempts to generalise Minkowski's theorem to non-symmetric convex bodies (cf. [27, pp. 52]). Here we just want to mention a still open and quite fascinating conjecture due to Ehrhart [22]

Conjecture 2.4 (Ehrhart)**.** *Let $K \subset \mathbb{R}^n$ be a convex body whose centre of gravity is the origin. If $\operatorname{vol}(K) \geq (n+1)^n/n!$ then K contains a non-trivial lattice point.*

This inequality would be tight as the n-dimensional simplex $-\mathbf{1} + (n+1)T_n$ shows, where $\mathbf{1} = \sum_{i=1}^n e_i$ denotes the all 1-vector, e_i the i-th unit vector and $T_n = \operatorname{conv}\{0, e_1, \ldots, e_{n+1}\}$ the standard simplex in $\mathbb{R}^n$ of volume $1/n!$. So far the conjecture has only be proven in the plane and for special 3-dimensional convex bodies by Ehrhart [22].

3. Intrinsic volumes

Let B^i be the i-dimensional unit ball whose i-dimensional volume will be denoted by κ_i. For a convex body $K \subset \mathbb{R}^n$ the outer parallel body at distance $\epsilon > 0$ is given by $K + \epsilon\, B^n$, i.e., it consists of all points whose (Euclidean) distance to K is at most ϵ. The volume of $K + \epsilon\, B^n$ is a polynomial of degree n in ϵ, the so called Steiner-polynomial (cf. [48, pp. 197])

$$\operatorname{vol}(K + \epsilon B^n) = \sum_{i=0}^{n} \epsilon^{n-i}\, \kappa_{n-i}\, \mathrm{V}_i(K). \tag{3.1}$$

The coefficients $\mathrm{V}_i(K)$, $i = 0, \ldots, n$, are the so called intrinsic volumes. They are normalised Minkowskian Quermassintegrals and they were introduced by Peter McMullen [39]. In particular, we have $\mathrm{V}_n(K) = \operatorname{vol}(K)$, $\mathrm{V}_{n-1}(K)$ is one half of the surface area $\mathrm{F}(K)$ and $\mathrm{V}_0(K) = 1$.

Since the volume functional itself is an intrinsic volume and in view of (1.3) one may ask for further inequalities involving the successive minima and other intrinsic volumes. Concerning a lower bound we have the following generalisation of (1.3) [56]

Theorem 3.1 (W.). *Let $K \in \mathcal{K}_0^n$. Then for $1 \le m \le n$*

$$\frac{1}{m!} \prod_{i=1}^{m} \frac{2}{\lambda_i(K)} \le \mathrm{V}_m(K).$$

This inequality is tight and for $m = n$ we obtain the lower bound in (1.3). In general, $\lambda_1(K) \cdot \ldots \cdot \lambda_m(K) \cdot \mathrm{V}_m(K)$ can not be bounded from above by a constant. Hence in order to find an extension of the upper bound in (1.3) to other intrinsic volumes a different type of inequality is needed. Here we have [28]

Theorem 3.2 (H.). *Let $K \in \mathcal{K}_0^n$. Then for $0 \le m \le n-1$*

$$\mathrm{vol}(K) \le \mathrm{V}_m(K) \prod_{i=m+1}^{n} \frac{2}{\lambda_i(K)}.$$

Again this inequality is best possible and for $m = 0$ we get Minkowski's upper bound in (1.3). Of particular interest and simplicity is the case $m = n-1$ which says that

$$\lambda_n(K) \le \frac{\mathrm{F}(K)}{\mathrm{vol}(K)}. \tag{3.2}$$

So for λ bigger than the ratio of surface area to volume the dilate $\lambda\, K$ contains n-linearly interior lattice points. Finally we want to remark that it is an open problem to generalise the theorems above to arbitrary lattices (cf. [26]).

4. Lattice point enumerator

By definition the successive minima measure or reflect a certain lattice point property of a 0-symmetric convex body. Hence it is quite natural to look for bounds on the lattice point enumerator $\mathrm{G}(K) = \#(K \cap \mathbb{Z}^n)$ in terms of the successive minima. A result in this spirit is again due to Minkowski who proved [40, p. 79]

Theorem 4.1 (Minkowski). *Let $K \in \mathcal{K}_0^n$ with $\lambda_1(K) \ge 1$, i.e., $\mathrm{int}(K) \cap \mathbb{Z}^n = \{0\}$. Then*

$$\mathrm{G}(K) \le 3^n.$$

The cube C_n shows that the bound can not be improved in general. Minkowski also proved a sharper bound of $2^{n+1} - 1$ for the class of strictly 0-symmetric convex bodies, but for simplification we will deal only with the general case and refer to [27, p. 63] for details. In [11] the above result was embedded in a more general inequality, namely

Theorem 4.2 (Betke,H.,W.). *Let $K \in \mathcal{K}_0^n$. Then*

$$\mathrm{G}(K) \le \left\lfloor \frac{2}{\lambda_1(K)} + 1 \right\rfloor^n.$$

This inequality may be considered as a lattice point analogue of Minkowski's first Theorem 1.1. Roughly speaking, we have only replaced the volume by the lattice point enumerator. Moreover, since K is a Jordan-measurable set we have

$$\lim_{m\to\infty} \frac{\mathrm{G}(m\,K)}{\mathrm{vol}(m\,K)} = 1, \tag{4.1}$$

and thus Theorem 4.2 implies Minkowski's first theorem. By the same reasoning the following lower bound proved in [11] is a generalisation of the lower bound in Minkowski's second Theorem 1.2

$$\left(1 - \frac{\lambda_1(K)}{2}\right) \frac{1}{n!} \prod_{i=1}^{n} \frac{2}{\lambda_i(K)} \leq \mathrm{G}(K).$$

Here we have additionally to assume that $\lambda_1(K) \leq 1$.

Of course, the most challenging problem is to give an analogue of the upper bound in Minkowski's second theorem but for that problem, unfortunately, we only have a conjecture [11]

Conjecture 4.3 (Betke, H., W.)**.** *Let* $K \in \mathcal{K}_0^n$. *Then*

$$\mathrm{G}(K) \leq \prod_{i=1}^{n} \left\lfloor \frac{2}{\lambda_i(K)} + 1 \right\rfloor.$$

As in Minkowski's upper bound we have equality for boxes with facets parallel to the coordinate axes (cf. (1.4)). So far this conjecture has only been verified in the 2-dimensional case [11] and in the general case it is only known that [29]

$$\mathrm{G}(K) \leq 2^{n-1} \prod_{i=1}^{n} \left\lfloor \frac{2}{\lambda_i(K)} + 1 \right\rfloor.$$

Since the conjecture would imply Minkowski's upper bound in (1.3) it seems to be a rather hard problem.

Now we relax a little bit the right hand side in the conjecture and set

$$\mathrm{L}(K) = \prod_{i=1}^{n} \left(\frac{2}{\lambda_i(K)} + 1\right).$$

Since the successive minima are homogeneous of degree -1 we see that for any positive number μ

$$\mathrm{L}(\mu\,K) = \prod_{i=1}^{n} \left(\frac{2}{\lambda_i(K)}\,\mu + 1\right) \tag{4.2}$$

is a polynomial of degree n in μ and Conjecture 4.3 relates this polynomial with the lattice point enumerator of a 0-symmetric convex body. There is another and quite famous polynomial related to the number of lattice points which will be discussed in the next section.

5. Ehrhart polynomials

In the following we will study the number of lattice points contained in lattice polytopes, which in our setting are polytopes whose vertices belong to the integral lattice $\mathbb{Z}^n$. The space of all n-dimensional lattice polytopes in $\mathbb{R}^n$ is denoted by $\mathcal{P}^n$. In 1899 G. Pick [44] found the following beautiful identity for lattice polygons

Theorem 5.1 (Pick). *Let $P \in \mathcal{P}^2$. Then*

$$\mathrm{G}(P) = \mathrm{vol}(P) + \frac{1}{2}\mathrm{G}(\mathrm{bd}P) + 1,$$

where bd() denotes the boundary. The statement is also true for non-selfintersecting (and not necessarily convex) polygons. The proof of this theorem relies heavily, as many other results on lattice points in the plane, on the 2-dimensional fact that the area of any lattice triangle whose vertices are the only lattice points is bounded. In fact, the volume is always $1/2$. In higher dimensions the so called Reeve simplices [46]

$$R_n(m) = \mathrm{conv}\{0, e_1, \dots, e_{n-1}, m\sum\nolimits_{i=1}^{n} e_i\},\ m \in \mathbb{N}, \tag{5.1}$$

show that the volume of such a simplex may not be bounded at all.

Recently, M. Ram Murty and Nithum Thain [43] gave a nice proof of Pick's theorem via Minkowski's theorem. Since for any integer k the dilate kP is again a lattice polygon Pick's theorem immediately implies

$$\begin{aligned}
\mathrm{G}(k\,P) &= \mathrm{vol}(P)\,k^2 + \frac{1}{2}\mathrm{G}(\mathrm{b}d P)\,k + 1,\\
\mathrm{G}(\mathrm{int}(k\,P)) &= \mathrm{vol}(P)\,k^2 - \frac{1}{2}\mathrm{G}(\mathrm{b}d P)\,k + 1.
\end{aligned}$$

In the years 1962-1968 E. Ehrhart [23, 24, 25] generalised these identities to all dimensions

Theorem 5.2 (Ehrhart). *Let $P \in \mathcal{P}^n$ and $k \in \mathbb{N}$. Then*

$$\begin{aligned}
\mathrm{G}(k\,P) &= \sum_{i=0}^{n} \mathrm{G}_i(P)\,k^i,\\
\mathrm{G}(\mathrm{int}(k\,P)) &= (-1)^n \sum_{i=0}^{n} \mathrm{G}_i(P)\,(-k)^i,
\end{aligned}$$

where the coefficients $\mathrm{G}_i(P)$ depends only on P.

The first identity is known as the Ehrhart polynomial and the second one is called Ehrhart's reciprocity law. Two of the $n+1$ coefficients $\mathrm{G}_i(P)$ are obvious, namely, $\mathrm{G}_0(P) = 1$ and $\mathrm{G}_n(P) = \mathrm{vol}(P)$. Also the second leading coefficient admits a simple geometric interpretation via the facets $F_1, \dots, F_m$ of the lattice polytope. Here we have (cf. [24])

$$\mathrm{G}_{n-1}(P) = \frac{1}{2}\sum_{i=1}^{m} \frac{\mathrm{vol}_{n-1}(F_i)}{\det(\mathrm{aff}F_i \cap \mathbb{Z}^n)}, \tag{5.2}$$

where $\mathrm{vol}_{n-1}()$ denotes the $(n-1)$-dimensional volume and $\det(\mathrm{aff}\, F_i \cap \mathbb{Z}^n)$ denotes the determinant of the $(n-1)$-dimensional sublattice of $\mathbb{Z}^n$ contained in the affine hull of the facet F_i. So $\mathrm{G}_{n-1}(P)$ may be regarded as the normalised surface area of P with respect to $\mathbb{Z}^n$ and, obviously we have $\mathrm{G}_{n-1}(P) \leq (1/2)\,\mathrm{F}(K)$.

All other coefficients $\mathrm{G}_i(P)$, $1 \leq i \leq n-2$, have no such direct geometric meaning, except for special classes of polytopes (cf., e.g., [9, 10, 20, 32, 38, 37, 42, 45]). Since its discovery the Ehrhart polynomial and its coefficients play an essential role in discrete geometry, geometry of numbers and combinatorics (cf., e.g., [26, 27, 31, 36, 53]). For instance, in [12] Betke and Kneser showed that the coefficients form a basis of all additive and unimodular invariant functionals on the space $\mathcal{P}^n$. Stanley studied the Hilbert series of an integral polytope and proved in this context his famous non-negativity theorem [52]. For representations of $\mathrm{G}_i(P)$ in terms of Todd classes of a toric variety associated with P we refer to [7] and the references within. Based on Barvinok's methods for counting lattice points (cf., e.g., [5, 6]), De Loera et al. developed an efficient algorithm for calculating the coefficients of the Ehrhart polynomial (cf. [18, 19]).

Before we present a relation between the coefficients of the Ehrhart polynomial and the successive minima we give some examples of Ehrhart polynomials. For the 3-dimensional Reeve simplex $R_3(m)$ (cf. (5.1)) we obtain by calculating the lattice points in $k\,R_3(m)$, $k = 1, 2, 3$, that

$$\mathrm{G}(k\,R_3(m)) = \frac{m}{6}\,k^3 + k^2 + \frac{12-m}{6}\,k + 1.$$

In particular, this shows that some of the G_i's might be negative. For the simplex $T_n = \mathrm{conv}\{0, e_1, \ldots, e_n\}$ and the cross-polytope $C_n^\star$ one easily finds

$$\mathrm{G}(k\,T_n) = \binom{n+k}{n}, \qquad \mathrm{G}(k\,C_n^*) = \sum_{i=0}^{n} 2^i \binom{n}{i}\binom{k}{i}.$$

If Q is a lattice box defined via integers α_i (cf. (1.4)) we have

$$\mathrm{G}(k\,Q) = \prod_{i=1}^{n} (1 + 2\,\alpha_i\,k) \tag{5.3}$$

and, in particular, for the cube C_n we obtain

$$\mathrm{G}_i(C_n) = 2^{n-i}\binom{n}{i}.$$

In recent years the Ehrhart polynomial was not only regarded as a polynomial for integers k, but as a formal polynomial of a complex variable $s \in \mathbb{C}$ (cf. [8, 47, 57]). Therefore, for $P \in \mathcal{P}^n$ and $s \in \mathbb{C}$ we set

$$\mathrm{G}(s, P) = \sum_{i=0}^{n} \mathrm{G}_i(P)\,s^i = \prod_{i=1}^{n}\left(1 + \frac{s}{\gamma_i(P)}\right),$$

where $-\gamma_i(P) \in \mathbb{C}$, $1 \leq i \leq n$, are the roots of the Ehrhart polynomial $\mathrm{G}(s, P)$. So $\mathrm{G}_i(P)$ is the i-th elementary symmetric polynomial of $1/\gamma_1, \ldots, 1/\gamma_n$ and, in particular, we have

$$\mathrm{vol}(P) = \mathrm{G}_n(P) = \prod_{i=1}^{n} \frac{1}{\gamma_i(P)} \quad \text{and} \quad \mathrm{G}_{n-1}(P) = \sum_{j=1}^{n} \prod_{i \neq j} \frac{1}{\gamma_i(P)}. \tag{5.4}$$

For bounds and further properties of the roots of the Ehrhart polynomial we refer to [8, 30]. Here we are interested in relations between $\gamma_i(P)$ and Minkowski's successive minima. For a lattice boxes Q as considered in (5.3) we obviously have

$$\frac{\lambda_i(Q)}{2} = \frac{1}{2\,\alpha_i} = \gamma_i(Q),\ 1 \leq i \leq n.$$

In terms of the roots of the Ehrhart polynomial (cf. (5.4)) we can rewrite Minkowski's second Theorem 1.2 for $P \in \mathcal{P}^n \cap \mathcal{K}_0^n$ as

$$\left(\prod_{i=1}^{n} \frac{\lambda_i(P)}{2}\right)^{1/n} \leq \left(\prod_{i=1}^{n} \gamma_i(P)\right)^{1/n} \leq n!^{1/n} \left(\prod_{i=1}^{n} \frac{\lambda_i(P)}{2}\right)^{1/n}. \tag{5.5}$$

These inequalities between the geometric mean of the successive minima and the roots of the Ehrhart polynomial lead naturally to the question of further inequalities among the elementary symmetric functions of $\lambda_i(P)/2$ and $\gamma_i(P)$. so far the only other known relation is between the arithmetic means [30]

Theorem 5.3 (H.,Schürmann,W.). *Let $P \in \mathcal{P}^n \cap \mathcal{K}_0^n$. Then*

$$\frac{1}{n}\left(\sum_{i=1}^{n} \gamma_i(P)\right) \leq \frac{1}{n}\left(\sum_{i=1}^{n} \frac{\lambda_i(P)}{2}\right) \tag{5.6}$$

and the bound is tight.

An interesting feature of this inequality is the fact that it is tight for the cube C_n as well as for the cross polytope $C_n^\star$. However, there does not seem to be a lower bound on the arithmetic mean of $\gamma_i(P)$ by the arithmetic mean of $\lambda_i(P)$ (cf. [30]). In terms of the coefficients of the Ehrhart polynomial, Theorem 5.3 is equivalent to the more geometric statement (cf. (5.4))

$$\frac{\mathrm{G}_{n-1}(P)}{\mathrm{vol}(P)} \leq \frac{1}{2} \sum_{i=1}^{n} \lambda_i(P). \tag{5.7}$$

In comparison with (3.2) we see that the ratio of surface area to volume can be bounded from below by the successive minima whereas the ratio of the lattice surface area to the volume can be bounded form above. From (5.7) and the upper bound in (1.3) we get

Corollary 5.4. *Let $P \in \mathcal{P}_n \cap \mathcal{K}_0^n$. Then*

$$\mathrm{G}_{n-1}(P) \leq \sum_{j=1}^{n} \prod_{i \neq j} \frac{2}{\lambda_i(P)}.$$

This corollary brings us back to the polynomial

$$\mathrm{L}(\mu\,K)=\prod_{i=1}^{n}\left(\frac{2}{\lambda_i(K)}\,\mu+1\right)$$

introduced in (4.2). Denoting the coefficients of this polynomial by $\mathrm{L}_i(K)$ the right hand side in the corollary above is exactly $\mathrm{L}_{n-1}(P)$ and the leading coefficient is equal to $\prod_{i=1}^{n}2/\lambda_i(P)$. Thus, by the upper bound in Minkowski's second theorem and Corollary 5.4 we obtain for $P\in\mathcal{P}_n\cap\mathcal{K}_0^n$ the two inequalities

$$\mathrm{G}_n(P)\leq\mathrm{L}_n(P)\text{ and }\mathrm{G}_{n-1}(P)\leq\mathrm{L}_{n-1}(P).$$

These two relations also support the conjectured inequality $G(K)\leq\mathrm{L}(K)$ for $K\in\mathcal{K}_0^n$ which follows from the stronger Conjecture 4.3. Observe, that it is enough to prove $\mathrm{G}(K)\leq\mathrm{L}(K)$ for 0-symmetric lattice polytopes. It seems to be quite unlikely, however, to prove $G(P)\leq L(P)$ via the associate polynomials in a coefficient-wise way. Nevertheless for special polytopes this approach may work. For instance, let P be a 0-symmetric n-dimensional lattice polytope with $\mathrm{int}(P)\cap\mathbb{Z}^n=\{0\}$ and thus $\lambda_i(P)=1$, $1\leq i\leq n$. Then $\mathrm{L}_i(P)=\binom{n}{i}2^i$ and $\mathrm{G}_i(P)\leq\mathrm{L}_i(P)$ is equivalent to a conjecture of Wills [56]

Conjecture 5.5 (W.). *Let* $P\in\mathcal{P}_n\cap\mathcal{K}_0^n$ *such that* $\mathrm{int}(P)\cap\mathbb{Z}^n=\{0\}$. *Then*

$$\mathrm{G}_i(P)\leq\binom{n}{i}2^i,\quad 1\leq i\leq n.$$

Of course, the case $i=n$ follows from Minkowski's first Theorem 1.1.

6. Best simultaneous Diophantine approximations

Here we want to extend Minkowski's theorems on successive minima to periodic lattices. As in the previous sections we will present the results only with respect to the integral lattice, but as before they can easily be generalised to aribitrary lattices. For $\alpha\in\mathbb{R}^n$ and an integer $Q\in\mathbb{N}_{\geq 0}$ we set

$$\mathbb{Z}^n(\alpha,Q)=\mathbb{Z}^n\cup(\alpha+\mathbb{Z}^n)+(2\,\alpha+\mathbb{Z}^n)\cup\cdots\cup(Q\,\alpha+\mathbb{Z}^n),$$

where we always assume that $k\alpha\notin\mathbb{Z}^n$ for $1\leq k\leq Q$. In order to allow the case $\alpha\in\Lambda$ we admit $Q=0$. $\mathbb{Z}^n(\alpha,Q)$ is called a periodic lattice. Next we define for $K\in\mathcal{K}_0^n$ and $1\leq i\leq n$

$$\lambda_i(K,\mathbb{Z}^n(\alpha,Q))=\min\{\lambda>0:\dim(\lambda\,K\cap\mathbb{Z}^n(\alpha,Q))\geq i\}.$$

Obviously, for $Q=0$ we just have Minkowski's successive minima introduced in Section 1. Denoting by $|x|_K=\min\{\lambda\geq 0: x\in\lambda\,K\}$ the norm of $x\in\mathbb{R}^n$ induced by K, the first successive minimum of the series above can be expressed by

$$\lambda_1(K,\mathbb{Z}^n(\alpha,Q))=\min\{|q\alpha-z|_K>0: q\in\{0,\dots,Q\},\,z\in\mathbb{Z}^n\}.$$

So it measures the quality of a best approximation of α by a rational vector whose common denominator is bounded by Q. This functional has been studied from various respects. For instance, for $n=1$ and based on continued fractions, Klein [34] gave a geometric interpretation of such a "best approximation point" as a vertex of an associated 2-dimensional Klein polyhedron. Davenport and Mahler [17] proved that there exist infinitely many points $(q,z)^\intercal\in\mathbb{Z}^3$, $z\in\mathbb{Z}^2$, such that $|q\alpha - z|^2_{B^2} \leq (2/\sqrt{23})/q$ and the constant on the right hand side is best possible. The first who embedded $\lambda_1(K, \mathbb{Z}^n(\alpha, Q))$ in a series of functionals were W.B. Jurkat [33] and W. Kratz [35]. Their functionals are closely linked to the $\lambda_i(K, \mathbb{Z}^n(\alpha, Q))$ but defined in a space of dimension $n+1$.

In order to present Minkowski-type inequalities for the $\lambda_i(K, \mathbb{Z}^n(\alpha, Q))$ we also need the notation of the density of a densest (not necessarily lattice) packing of K which will be denoted by $\delta^*(K)$ (cf [27, pp. 223]). Then we have the following analogues to Minkowski's theorems [2]

Theorem 6.1 (Aliev, H.)**.** *Let $K \in \mathcal{K}_0^n$, $\alpha \in \mathbb{R}^n$ and $Q \in \mathbb{N}_{\geq 0}$ such that $k\alpha \notin \Lambda$ for $1 \leq k \leq Q$. Then with $\lambda_i = \lambda_i(K, \mathbb{Z}^n(\alpha, Q))$ we have*

$$\begin{aligned} &\text{i)} \quad (\lambda_1)^n \mathrm{vol}(K) \leq \delta^*(K)\, 2^n\, frac1Q+1, \\ &\text{ii)} \quad \gamma(\alpha, Q, n)\frac{2^n}{n!} \leq \lambda_1 \cdot \ldots \cdot \lambda_n \,\mathrm{vol}(K) \leq 2^n \,\frac{1}{Q+1}, \end{aligned}$$

where $\gamma(\alpha, Q, n)$ is a certain constant depending on α, Q and n.

Inequalities of that type give us information on the quality of the simultaneous approximation of a vector by a system of linearly independent rational vectors whose common denominators are bounded.

For details on the constant $\gamma(\alpha, Q, n)$ in the lower bound of Theorem 6.1 ii) we refer to the paper [2], but we want to remark that in the case $(Q+1)\alpha \in \mathbb{Z}^n$, i.e., $\mathbb{Z}^n(\alpha, Q)$ is a lattice, we have $\gamma(\alpha, Q, n) \geq 1/(Q+1)$. Thus Theorem 6.1 may be regarded as an extension of Minkowski's inequalities (1.2) and (1.3) to periodic lattices. Actually, the first inequality i) looks quite similar to (2.1), but since the inequality is valid for a more general structure than lattices we have to replace the density of densest lattice packing in (2.1) by $\delta^\star(K)$.

If $K = C_n$ then $|\cdot|_K$ is the maximum norm and Theorem 6.1 i) says that for any $\alpha \in \mathbb{R}^n$ there exists a $z \in \mathbb{Z}^n$ and $q \in \{1, \ldots, Q\}$ such that

$$\left|\alpha_i - \frac{z_i}{q}\right| < \frac{Q^{-1/n}}{q}, \ 1 \leq i \leq n.$$

This is Dirichlet's classical theorem on simultaneous Diophantine approximation.

Quite recently Aliev and Gruber [1] also gave a lower estimate for the quantity $\lambda_1(K, \mathbb{Z}^n(\alpha, Q))$ which, with the notation of Theorem 6.1, can be formulated as

Theorem 6.2 (Aliev, Gruber)**.** *For every $\epsilon > 0$ there exists an $\alpha \in \mathbb{R}^n$ and $Q \in \mathbb{N}$ such that*

$$(\lambda_1)^n \,\mathrm{vol}(K) > \delta(K)\, 2^n \frac{1}{Q}.$$

In fact, they proved this theorem not only for 0-symmetric convex bodies, but for any bounded star body K.

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